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A FULLY NONLINEAR EQUATION ON FOUR-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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Abstract

We present a conformal deformation involving a fully nonlinear equation in dimension 4, starting with a metric of positive scalar curvature. Assuming a certain conformal invariant is positive, one may deform from positive scalar curvature to a stronger condition involving the Ricci tensor. A special case of this deformation provides an alternative proof to the main result in Chang, Gursky & Yang, 2002. We also give a new conformally invariant condition for positivity of the Paneitz operator, generalizing the results in Gursky, 1999. From the existence results in Chang & Yang, 1995, this allows us to give many new examples of manifolds admitting metrics with constant Q-curvature.

1. Introduction

Let (M, g) denote a closed, 4-dimensional Riemannian manifold, and let $Y[g]$ denote the Yamabe invariant of the conformal class $[g]$:

$$Y[g] \equiv \inf_{\widetilde{g}\in[g]} \mathrm{Vol}\,(\widetilde{g})^{-1/2} \int_M R_{\widetilde{g}} d\mathrm{vol}_{\widetilde{g}}, \tag{1.1}$$

where $R_{\widetilde{g}}$ denotes the scalar curvature. Another important conformal invariant is

$$\mathcal{F}_2([g]) \equiv \int_M \left(-\frac{1}{2} |\,\mathrm{Ric}_g|^2 + \frac{1}{6} R_g^2 \right) d\mathrm{vol}_g, \tag{1.2}$$

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where Ric_g is the Ricci tensor. By the Chern-Gauss-Bonnet formula ([4]),

$$8\pi^2\chi(M) = \int_M |W_g|^2 d\mathrm{vol}_g + \mathcal{F}_2([g]). \tag{1.3}$$

Thus, the conformal invariance of $\mathcal{F}_2$ follows from the well-known (pointwise) conformal invariance of the Weyl tensor W_g (see [13]).

Define the tensor

$$A_g^t = \frac{1}{2}\Big(\mathrm{Ric}_g - \frac{t}{6}R_g g\Big). \tag{1.4}$$

Note that for $t = 1$, A_g^1 is the classical Schouten tensor ([13]). Let $\sigma_2(g^{-1}A_g^t)$ denote the second elementary symmetric function of the eigenvalues of $g^{-1}A_g^t$, viewed as an endomorphism of the tangent bundle. Then a simple calculation gives

$$\mathcal{F}_2([g]) = 4\int_M \sigma_2(g^{-1}A_g^1) d\mathrm{vol}_g. \tag{1.5}$$

Our main result is the following:

Theorem 1.1. *Let (M, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If*

$$\mathcal{F}_2([g]) + \frac{1}{6}(1-t_0)(2-t_0)(Y[g])^2 > 0, \tag{1.6}$$

for some $t_0 \leq 1$, then there exists a conformal metric $\widetilde{g} = e^{-2u}g$ with $R_{\widetilde{g}} > 0$ and $\sigma_2(A_{\widetilde{g}}^{t_0}) > 0$ pointwise. This implies the pointwise inequalities

$$(t_0 - 1)R_{\widetilde{g}}\widetilde{g} < 2\,\mathrm{Ric}_{\widetilde{g}} < (2-t_0)R_{\widetilde{g}}\widetilde{g}. \tag{1.7}$$

As applications of Theorem 1.1, we consider two different values of t_0. When $t_0 = 1$, we obtain a different proof of the following result in [8]:

Corollary 1.1. *Let (M, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If $\mathcal{F}_2([g]) > 0$, then there exists a conformal metric $\widetilde{g} = e^{-2u}g$ with $R_{\widetilde{g}} > 0$ and $\sigma_2(\widetilde{g}^{-1}A_{\widetilde{g}}^1) > 0$ pointwise. In particular, the Ricci curvature of $\widetilde{g}$ satisfies*

$$0 < 2\,\mathrm{Ric}_{\widetilde{g}} < R_{\widetilde{g}}\widetilde{g}.$$

The proof in [8] involved regularization by a fourth-order equation and relied on some delicate integral estimates. By contrast, the proof of Theorem 1.1 seems more direct, and depends on general *a priori* estimates for fully nonlinear equations developed in [30], [20], [27], and [23].

Our second application is to the spectral properties of a conformally invariant differential operator known as the *Paneitz operator*. Let δ denote the L^2-adjoint of the exterior derivative d; then the Paneitz operator is defined by

$$P_g\phi = \Delta^2\phi + \delta\Big(\frac{2}{3}R_g g - 2\operatorname{Ric}_g\Big)d\phi. \tag{1.8}$$

The Paneitz operator is conformally invariant, in the sense that if $\widetilde{g} = e^{-2u}g$, then

$$P_{\widetilde{g}} = e^{4u}P_g. \tag{1.9}$$

Since the volume form of the conformal metric $\widetilde{g}$ is $d\mathrm{vol}_{\widetilde{g}} = e^{-4u}d\mathrm{vol}_g$, an immediate consequence of (1.9) is the conformal invariance of the Dirichlet energy

$$\langle P_{\widetilde{g}}\phi, \phi\rangle_{L^2(M,\widetilde{g})} = \langle P_g\phi, \phi\rangle_{L^2(M,g)}.$$

In particular, positivity of the Paneitz operator is a conformally invariant property, and clearly the kernel is invariant as well.

To appreciate the geometric significance of the Paneitz operator we need to define the associated Q-*curvature*, introduced by Branson:

$$Q_g = -\frac{1}{12}\Delta R_g + 2\sigma_2(g^{-1}A_g^1). \tag{1.10}$$

Under a conformal change of metric $\widetilde{g} = e^{-2u}g$, the Q-curvature transforms according to the equation

$$-Pu + 2Q_g = 2Q_{\widetilde{g}}e^{-4u}, \tag{1.11}$$

see, for example, [5]. Note that

$$\int_M Q_g d\mathrm{vol}_g = \frac{1}{2}\mathcal{F}_2([g]), \tag{1.12}$$

so the integral of the Q-curvature is conformally invariant.

The Q-curvature and Paneitz operator have become important objects of study in the geometry of four-manifolds, and play a role in such diverse topics as Moser-Trudinger inequalities ([3], [6]), compactification of complete conformally flat manifolds ([9]), twistor theory ([11]), gauge choices for Maxwell's equations ([12]), and most recently in the study of conformally compact AHE manifolds ([15], [18]).

Our interest here is in the spectral properties of the Paneitz operator and the related question of the existence of metrics with constant Q-curvature. The most general work on this subject was done by Chang and Yang [10], who studied the problem of constructing conformal metrics with constant Q-curvature by variational methods. They considered the functional

$$F[\phi] = \langle P_g \phi, \phi \rangle - 4 \int_M Q\phi d\text{vol} - \left(\int_M Q d\text{vol} \right) \log \int_M e^{-4\phi} d\text{vol}, \tag{1.13}$$

and analyzed the behavior of a minimizing sequence. Of course, it is not clear *a priori* that F is even bounded from below. Indeed, if the Paneitz operator has a negative eigenvalue and the conformal invariant (1.12) is positive, then Chang and Yang showed that $\inf F = -\infty$ (see [10], p. 177). For example, take a compact surface Σ of curvature -1 with first eigenvalue $\lambda_1(-\Delta) << 1$. Then the product manifold $M = \Sigma \times \Sigma$ will have $\lambda_1(P) < 0$ and $\int Q d\text{vol} > 0$.

Chang and Yang also pointed out the connection between the conformal invariant (1.12) and the best constant in the inequality of Adams [1], another key point for establishing the $W^{2,2}$ compactness of a minimizing sequence. Combining these observations, they proved:

Theorem 1.2 ([10]). *Let (M, g) be a compact 4-manifold. Assume:*

(i) *The Paneitz operator P_g is nonnegative with* $\operatorname{Ker} P = \{\text{constants}\}$.

(ii) *The conformal invariant (1.12) is strictly less than the value attained by the round sphere.*

Then there exists a minimizer of F, which satisfies (1.11) with $Q_{\tilde{g}} =$ constant.

Subsequently, the first author proved that any four-manifold of positive scalar curvature which is not conformally equivalent to the sphere already satisfies the second assumption of Chang-Yang. In addition:

Theorem 1.3 ([21]). *Let (M, g) be a compact 4-manifold. If the scalar curvature of g is nonnegative and $\int Q d\mathrm{vol} \geq 0$, then the Paneitz operator is positive and* $\operatorname{Ker} P = \{\text{constants}\}$.

Because of the example of Chang-Yang, it is clear that one cannot relax the condition on the scalar curvature in the above theorem. On the other hand, the positivity of the conformal invariant (1.12) is a rather strong assumption. For example, if the scalar curvature is strictly positive, then the positivity of (1.12) implies the vanishing of the first Betti number of M (see [22]). Thus, for example, the manifold $N\#(S^1 \times S^3)$ can not admit a metric of positive scalar curvature with $\int Q d\mathrm{vol} > 0$.

It is interesting to note that the positivity of the Paneitz operator was studied by Eastwood and Singer in [11] for reasons motivated by twistor theory. They constructed metrics on $k(S^3 \times S^1)$ for all $k > 0$ with $P \geq 0$ and $\operatorname{Ker} P = \{\text{constants}\}$. Since these manifolds have $\int Q d\mathrm{vol} < 0$, the Eastwood-Singer construction is in some respects complementary to the result of [21].

By combining Theorem 1.1 with $t_0 = 0$, and an integration by parts argument, we obtain a new criterion for the positivity of P:

Theorem 1.4. *Let (M, g) be a closed 4-dimensional Riemannian manifold with positive scalar curvature. If*

$$\int Q_g d\mathrm{vol}_g + \frac{1}{6}(Y[g])^2 > 0, \tag{1.14}$$

then the Paneitz operator is nonnegative, and $\operatorname{Ker} P = \{\text{constants}\}$. *Therefore, by the results in* [10], *there exists a conformal metric $\widetilde{g} = e^{-2u} g$ with $Q_{\widetilde{g}} = \text{constant}$.*

Since Theorem 1.4 allows the integral of the Q-curvature to be negative, we are able to use surgery techniques to construct many new examples of manifolds which admit metrics with constant Q. For example, we will show that

$$\begin{aligned}
N &= (S^2 \times S^2)\#k(S^1 \times S^3),\ k \leq 5,\\
N &= \mathbb{CP}^2\#k(S^1 \times S^3),\ k \leq 5,\\
N &= \mathbb{CP}^2\#k(\mathbb{RP}^4),\ k \leq 8,\\
N &= k(S^1 \times S^3)\#l(\mathbb{RP}^4),\ 2k + l \leq 9,
\end{aligned}$$

all admit metrics with constant Q. See Section 7 for additional examples.

For the proof of Theorem 1.1, we will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u}g$:

$$\sigma_2^{1/2}(g^{-1}A^t_{\tilde{g}}) = f(x)e^{2u}, \tag{1.15}$$

where $f(x) > 0$. We have the following formula for the transformation of A^t under a conformal change of metric $\tilde{g} = e^{-2u}g$:

$$A^t_{\tilde{g}} = A^t_g + \nabla^2 u + \frac{1-t}{2}(\Delta u)g + du \otimes du - \frac{2-t}{2}|\nabla u|^2 g. \tag{1.16}$$

Since $A^t = A^1 + \frac{1-t}{2}tr(A^1)g$, this formula follows easily from the standard formula for the transformation of the Schouten tensor (see [30]):

$$A^1_{\tilde{g}} = A^1_g + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g. \tag{1.17}$$

Using (1.16), we may write (1.15) with respect to the background metric g

$$\begin{aligned}\sigma_2^{1/2}\Big(g^{-1}\Big(\nabla^2 u + \frac{1-t}{2}(\Delta u)g \\ - \frac{2-t}{2}|\nabla u|^2 g + du \otimes du + A^t_g\Big)\Big) = f(x)e^{2u}.\end{aligned} \tag{1.18}$$

The choice of the right-hand side in (1.18) is quite flexible; the key requirement is simply that the exponent is a positive multiple of u. For negative exponents we lose the invertibility of the linearized equation and some key *a priori* estimates; see the proofs of Propositions 2.2 and 3.1.

Equation (1.18) was considered in our earlier work ([23]) in the context of negative curvature. Li and Li ([27]) used a similar path to prove existence of solutions of the conformally invariant equation involving more general symmetric functions of the eigenvalues, assuming the manifold is locally conformally flat. After completing this paper, we also received the preprint of Guan, Lin and Wang ([19]), where they used a similar deformation technique to obtain various results in the locally conformally flat setting.

We will use the continuity method: the assumption of positive scalar curvature will allow us to start at some $t = \delta$ very negative. We will then use the conformally invariant assumption (1.6) in Section 3, together with the Harnack inequality of [20] and [27] in Section 4, to prove compactness of the space of solutions. Existence of a solution at time t_0 and verification of the inequalities (1.7) will be proved in Section 5, thus completing the proof of Theorem 1.1. Theorem 1.4 will be proved in Section 6, and in Section 7 we give many new examples of manifolds admitting metrics with constant Q-curvature.

2. Ellipticity

In this section we will discuss the ellipticity properties of Equation (1.18).

Definition 1. Let $(\lambda_1, \ldots, \lambda_4) \in \mathbf{R}^4$. We view the second elementary symmetric function as a function on $\mathbf{R}^4$:

$$\sigma_2(\lambda_1, \ldots, \lambda_4) = \sum_{i<j} \lambda_i \lambda_j, \tag{2.1}$$

and we define

$$\Gamma_2^+ = \{\sigma_2 > 0\} \cap \{\sigma_1 > 0\}, \tag{2.2}$$

where $\sigma_1 = \lambda_1 + \cdots + \lambda_4$ denotes the trace.

For a symmetric linear transformation $A : V \to V$, where V is an n-dimensional inner product space, the notation $A \in \Gamma_2^+$ will mean that the eigenvalues of A lie in the corresponding set. We note that this notation also makes sense for a symmetric tensor on a Riemannian manifold. If $A \in \Gamma_2^+$, let $\sigma_2^{1/2}(A) = \{\sigma_2(A)\}^{1/2}$.

Definition 2. Let $A : V \to V$ be a symmetric linear transformation, where V is an n-dimensional inner product space. The *first Newton transformation* associated with A is

$$T_1(A) = \sigma_1(A) \cdot I - A. \tag{2.3}$$

Also, for $t \in \mathbf{R}$ we define the linear transformation

$$L^t(A) = T_1(A) + \frac{1-t}{2}\sigma_1(T_1(A)) \cdot I. \tag{2.4}$$

We note that if $A_s : \mathbf{R} \to \mathrm{Hom}(V, V)$, then

$$\frac{d}{ds}\sigma_2(A_s) = \sum_{i,j} T_1(A_s)_{ij} \frac{d}{ds}(A_s)_{ij}, \tag{2.5}$$

that is, the first Newton transformation is what arises from differentiation of σ_2.

Proposition 2.1.

(i) *The set* Γ_2^+ *is an open convex cone with vertex at the origin.*

(ii) *If the eigenvalues of* A *are in* Γ_2^+*, then* $T_1(A)$ *is positive definite. Consequently, for* $t \leq 1$*,* $L^t(A)$ *is also positive definite.*

(iii) *For symmetric linear transformations* $A \in \Gamma_2^+$*,* $B \in \Gamma_2^+$*, and* $s \in [0, 1]$*, we have the following inequality*

$$\{\sigma_2((1-s)A + sB)\}^{1/2} \geq (1-s)\{\sigma_2(A)\}^{1/2} + s\{\sigma_2(B)\}^{1/2}. \tag{2.6}$$

Proof. The proof of this proposition is standard, and may be found in [7] and [16]. q.e.d.

For $u \in C^2(M)$, we define

$$A_u^t = A_g^t + \nabla^2 u + \frac{1-t}{2}(\Delta u)g + du \otimes du - \frac{2-t}{2}|\nabla u|^2 g. \tag{2.7}$$

Proposition 2.2. *Let* $u \in C^2(M)$ *be a solution of*

$$\sigma_2^{1/2}(g^{-1}A_u^t) = f(x)e^{2u}, \tag{2.8}$$

for some $t \leq 1$ *with* $A_u^t \in \Gamma_2^+$*. Then the linearized operator at* u*,* $\mathcal{L}^t : C^{2,\alpha}(M) \to C^\alpha(M)$*, is invertible* $(0 < \alpha < 1)$*.*

Proof. We define

$$F_t[x, u, \nabla u, \nabla^2 u] = \sigma_2(g^{-1}A_u^t) - f(x)^2 e^{4u},$$

so that solutions of (2.8) are zeroes of F_t. We then suppose that $u \in C^2(M)$ satisfies $F_t[x, u, \nabla u, \nabla^2 u] = 0$, with $A_u^t \in \Gamma_2^+$. Define $u_s = u + s\varphi$, then

$$\begin{aligned} \mathcal{L}^t(\varphi) &= \frac{d}{ds}F_t[x, u_s, \nabla u_s, \nabla^2 u_s]\Big|_{s=0} \\ &= \frac{d}{ds}\Big(\sigma_2(g^{-1}A_{u_s}^t)\Big)\Big|_{s=0} - \frac{d}{ds}\left(f^2 e^{4u_s}\right)\Big|_{s=0}. \end{aligned} \tag{2.9}$$

From (2.5), we have (using the summation convention)

$$\frac{d}{ds}\Big(\sigma_2(g^{-1}A_u^t)\Big)\Big|_{s=0} = T_1(g^{-1}A_u^t)_{ij}\frac{d}{ds}\Big((g^{-1}A_{u_s}^t)_{ij}\Big)\Big|_{s=0}.$$

We compute

$$\frac{d}{ds}\Big((g^{-1}A^t_{u_s})\Big)\Big|_{s=0} = g^{-1}\Big(\nabla^2\varphi + \frac{1-t}{2}(\Delta\varphi)g - (2-t)\langle du, d\varphi\rangle g + 2du\otimes d\varphi\Big).$$

Therefore,

$$\frac{d}{ds}\Big(\sigma_2(g^{-1}A^t_{u_s})\Big)\Big|_{s=0} = T_1(g^{-1}A^t_u)_{ij}\Big\{g^{-1}\Big(\nabla^2\varphi + (1-t)(\Delta\varphi)(g/2) - (2-t)\langle du, d\varphi\rangle g + 2du\otimes d\varphi\Big)\Big\}_{ij}. \tag{2.10}$$

For the second term on the right-hand side of (2.9) we have

$$\frac{d}{ds}\left(f^2 e^{4u_s}\right)\Big|_{s=0} = 4f^2e^{4u}\varphi. \tag{2.11}$$

Combining (2.10) and (2.11), we conclude

$$\mathcal{L}^t(\varphi) = T_1(g^{-1}A^t_u)_{ij}\{g^{-1}(\nabla^2\varphi + (1-t)(\Delta\varphi)(g/2))\}_{ij} - 4f^2e^{4u}\varphi + \cdots \tag{2.12}$$

where $+\cdots$ denotes additional terms which are linear in $\nabla\varphi$. Using the definition of L^t in (2.4), we can rewrite the leading term of (2.12) and obtain

$$\mathcal{L}^t(\varphi) = L^t(g^{-1}A^t_u)_{ij}(g^{-1}\nabla^2\varphi)_{ij} - 4f^2e^{4u}\varphi + \cdots \tag{2.13}$$

For $t \leq 1$, Proposition 2.1 implies that $L^t(g^{-1}A^t_u)$ is positive definite, so $\mathcal{L}^t$ is elliptic. Since the coefficient of φ in the zeroth-order term of (2.13) is strictly negative, the lineariztion is furthermore invertible on the stated Hölder spaces (see [17]). q.e.d.

3. C^0 estimate

Throughout the sequel, (M, g) will be a closed 4-dimensional Riemannian manifold with positive scalar curvature. Since $R_g > 0$, there exists $\delta > -\infty$ so that A^δ_g is positive definite. For $t \in [\delta, 1]$, consider the path of equations

$$\sigma_2^{1/2}(g^{-1}A^t_{u_t}) = f(x)e^{2u_t}, \tag{3.1}$$

where $f(x) = \sigma_2^{1/2}(g^{-1}A^\delta_g) > 0$. Note that $u \equiv 0$ is a solution of (3.1) for $t = \delta$.

Proposition 3.1. *Let $u_t \in C^2(M)$ be a solution of (3.1) for some $\delta \leq t \leq 1$. Then $u_t \leq \overline{\delta}$, where $\overline{\delta}$ depends only upon g.*

Proof. From Newton's inequality $\frac{4}{\sqrt{6}}\sigma_2^{1/2} \leq \sigma_1$, so

$$\frac{4}{\sqrt{6}} f(x) e^{2u_t} \leq \sigma_1(g^{-1}A^t_{u_t}). \tag{3.2}$$

Let p be a maximum of u_t, then the gradient terms vanish at p, and $\Delta u_t \leq 0$, so by (1.16)

$$\begin{aligned}\frac{4}{\sqrt{6}} f(p) e^{2u_t(p)} &\leq \sigma_1(g^{-1}A^t_{u_t})(p) \\ &= \sigma_1(g^{-1}A^t_g) + (3-2t)\Delta u_t \\ &\leq \sigma_1(g^{-1}A^t_g).\end{aligned}$$

Since $t \geq \delta$, this implies $u_t \leq \overline{\delta}$. q.e.d.

Proposition 3.2. *Assume that for some $\delta \leq t \leq 1$,*

$$\mathcal{F}_2([g]) + \frac{1}{6}(1-t)(2-t)(Y[g])^2 = \lambda_t > 0. \tag{3.3}$$

If $u_t \in C^2(M)$ is a solution of (3.1) satisfying $\|\nabla u_t\|_{L^\infty} < C_1$, then $u_t > \underline{\delta}$, where $\underline{\delta}$ depends only upon g, C_1, and $\log \lambda_t$.

Proof. Using Lemma 24 in [30], we have

$$\begin{aligned}\sigma_2(A^t) &= \sigma_2\Big(A^1 + \frac{1-t}{2}\sigma_1(A^1)g\Big) \\ &= \sigma_2(A^1) + 3\frac{1-t}{2}\sigma_1(A^1)^2 + 6\Big(\frac{1-t}{2}\sigma_1(A^1)\Big)^2 \\ &= \sigma_2(A^1) + \frac{3}{2}(1-t)(2-t)\sigma_1(A^1)^2.\end{aligned}$$

Letting $\widetilde{g} = e^{-2u_t}g$,

$$\begin{aligned}e^{4u_t}f^2 = \sigma_2(g^{-1}A^t_{u_t}) &= \sigma_2(g^{-1}A^1_{u_t}) + \frac{3}{2}(1-t)(2-t)\big(\sigma_1(g^{-1}A^1_{u_t})\big)^2 \\ &= e^{-4u_t}\Big(\sigma_2(\widetilde{g}^{-1}A^1_{u_t}) + \frac{1}{24}(1-t)(2-t)R^2_{\widetilde{g}}\Big).\end{aligned}$$

Integrating this, we obtain

$$\begin{aligned}
&C' \int_M e^{4u_t} d\mathrm{vol}_g \\
&\geq \int_M f^2 e^{4u_t} d\mathrm{vol}_g \\
&= \int_M \sigma_2(\widetilde{g}^{-1} A^1_{u_t}) e^{-4u_t} d\mathrm{vol}_g + \frac{1}{24}(1-t)(2-t) \int_M R^2_{\widetilde{g}} e^{-4u_t} d\mathrm{vol}_g \\
&= \int_M \sigma_2(\widetilde{g}^{-1} A^1_{\widetilde{g}}) d\mathrm{vol}_{\widetilde{g}} + \frac{1}{24}(1-t)(2-t) \int_M R^2_{\widetilde{g}} d\mathrm{vol}_{\widetilde{g}},
\end{aligned}$$

where $C' > 0$ is chosen so that $f^2 \leq C'$.

Lemma 3.1. *For any metric $g' \in [g]$, we have*

$$\int_M R^2_{g'} d\mathrm{vol}_{g'} \geq (Y[g])^2. \tag{3.4}$$

Proof. From Hölder's inequality,

$$\int_M R_{g'} d\mathrm{vol}_{g'} \leq \left\{ \int_M R^2_{g'} d\mathrm{vol}_{g'} \right\}^{1/2} \cdot \left\{ \mathrm{Vol}\,(g') \right\}^{1/2}. \tag{3.5}$$

Since g has positive scalar curvature, $Y[g] > 0$, so the left-hand side of (3.5) must be positive. We then obtain

$$(Y[g])^2 \leq \left(\mathrm{Vol}\,(g')^{-1/2} \int_M R_{g'} d\mathrm{vol}_{g'} \right)^2 \leq \int_M R^2_{g'} d\mathrm{vol}_{g'}.$$

q.e.d.

Using the lemma, and the conformal invariance of $\mathcal{F}_2$, we obtain

$$C' \int_M e^{4u_t} d\mathrm{vol}_g \geq \frac{1}{4}\mathcal{F}_2([g]) + \frac{1}{24}(1-t)(2-t)(Y[g])^2 = \frac{1}{4}\lambda_t > 0. \tag{3.6}$$

This implies

$$\max u_t \geq \frac{1}{4} \log \lambda_t - C(g). \tag{3.7}$$

The assumption $|\nabla u_t| < C_1$ implies the Harnack inequality

$$\max u_t \leq \min u_t + C(C_1, g), \tag{3.8}$$

by simply integrating along a geodesic connecting points at which u_t attains its maximum and minimum. Combining (3.7) and (3.8) we obtain

$$\min u_t \geq \frac{1}{4}\log\lambda_t - C.$$

q.e.d.

4. Harnack inequality

We next have the following C^1 estimate for solutions of Equation (1.18).

Proposition 4.1. *Let u_t be a C^3 solution of (3.1) for some $\delta \leq t \leq 1$, satisfying $u_t < \overline{\delta}$. Then $\|\nabla u_t\|_{L^\infty} < C_1$, where C_1 depends only upon $\overline{\delta}$ and g.*

Remark. A Harnack inequality was proved for the conformally invariant equation for $t = 1$ in [20], and then extended to $t < 1$ in [27]. More specifically, in [27] was considered the equation

$$\sigma_k^{1/k}(sA^1 + (1-s)\sigma_1(A^1)g) = f(x)e^{-2u}. \tag{4.1}$$

The left-hand side is just a reparametrization of A^t, but (3.1) has a different right-hand side, so the Harnack inequality now depends on the *sup*. The differences are minor, but for convenience, we present an outline of the proof here, and also provide a simple direct proof which works for $t < 1$.

Proof. Consider the function $h = |\nabla u|^2$ (we will omit the subscript on u_t). Since M is compact, and h is continuous, we suppose the maximum of h occurs and a point $p \in N$. Take a normal coordinate system $(x^1, \ldots, x^n)$ at p, then $g_{ij}(p) = \delta_{ij}$, and $\Gamma^i_{jk}(p) = 0$, where $g = g_{ij}dx^i dx^j$, and Γ^i_{jk} is the Christoffel symbol (see [4]).

Locally, we may write h as

$$h = g^{lm}u_l u_m. \tag{4.2}$$

In a neighborhood of p, differentiating h in the x^i direction we have

$$\partial_i h = h_i = \partial_i(g^{lm}u_l u_m) = \partial_i(g^{lm})u_l u_m + 2g^{lm}\partial_i(u_l)u_m. \tag{4.3}$$

Since in a normal coordinate system, the first derivatives of the metric vanish at p, and since p is a maximum for h, evaluating (4.3) at p, we have

$$u_{li}u_l = 0. \tag{4.4}$$

Next we differentiate (4.3) in the x^j direction. Since p is a maximum, $\partial_j\partial_i h = h_{ij}$ is negative semidefinite, and we get (at p)

$$0 \gg h_{ij} = \frac{1}{2}\partial_j\partial_i g^{lm}u_l u_m + u_{lij}u_l + u_{li}u_{lj}. \tag{4.5}$$

We recall from Section 2 that

$$L^t_{ij} = T_{ij} + \frac{1-t}{2}\sum_l T_{ll}\delta_{ij}, \tag{4.6}$$

is positive definite, where T_{ij} means $(T_1(g^{-1}A^t_u))_{ij}$. We sum with (4.5) with L^t_{ij} to obtain the inequality

$$0 \geq \frac{1}{2}L^t_{ij}\partial_i\partial_j g^{lm}u_l u_m + L^t_{ij}u_{lij}u_l + L^t_{ij}u_{li}u_{lj}. \tag{4.7}$$

We next differentiate Equation (3.1) in order to replace the u_{lij} term with lower order terms. With respect to our local coordinate system, from (2.7) we have

$$\begin{aligned}(A^t_u)_{ij} = (A^t_g)_{ij} + u_{ij} - u_r\Gamma^r_{ij} + \frac{1-t}{2}\sum_k(u_{kk} - u_r\Gamma^r_{kk})g_{ij} + u_i u_j \\ - \frac{2-t}{2}(g^{r_1 r_2}u_{r_1}u_{r_2})g_{ij}.\end{aligned} \tag{4.8}$$

At the point p, this simplifies to

$$(A^t_u)_{ij} = (A^t_g)_{ij} + u_{ij} + \frac{1-t}{2}\sum_k(u_{kk})g_{ij} + u_i u_j - \frac{2-t}{2}(|\nabla u|^2)\delta_{ij}. \tag{4.9}$$

Next we take m with $1 \leq m \leq n$, and differentiate (3.1) with respect to x^m in our local coordinate system:

$$\partial_m\left\{\sigma_2\big(g^{lj}(A^t_u)_{ij}\big)\right\} = \partial_m(f(x)^2 e^{4u}). \tag{4.10}$$

Differentiating and evaluating at p, we obtain

$$\begin{aligned}T_{ij}\Big(\partial_m(A_g^t)_{ij}+u_{ijm}-u_r\partial_m\Gamma_{ij}^r \\ +\frac{1-t}{2}\sum_k(u_{kkm}-u_r\partial_m\Gamma_{kk}^r)\delta_{ij}+2u_{im}u_j\Big) \\ =(\partial_m f^2)e^{4u}+4f^2e^{4u}u_m.\end{aligned}\tag{4.11}$$

Note that the third order terms in the above expression are

$$T_{ij}\Big(u_{ijm}+\frac{1-t}{2}\sum_k u_{kkm}\delta_{ij}\Big)=L_{ij}^t u_{ijm}.$$

Next we sum (4.11) with u_m, using (4.4) we have the following formula

$$\begin{aligned}L_{ij}^t u_m u_{ijm}+T_{ij}\Big(u_m\partial_m(A_g^t)_{ij} \\ -u_m u_r\partial_m\Gamma_{ij}^r-\frac{1-t}{2}\sum_k(u_r u_m\partial_m\Gamma_{kk}^r)\delta_{ij}\Big) \\ =u_m(\partial_m f^2)e^{4u}+4f^2e^{4u}|\nabla u|^2.\end{aligned}\tag{4.12}$$

Substituting (4.12) into (4.7), we arrive at the inequality

$$\begin{aligned}0\geq\frac{1}{2}L_{ij}^t\partial_i\partial_j g^{lm}u_l u_m+T_{ij}\Big(-u_m\partial_m(A_g^t)_{ij} \\ +u_m u_r\partial_m\Gamma_{ij}^r+\frac{1-t}{2}\sum_k(u_r u_m\partial_m\Gamma_{kk}^r)\delta_{ij}\Big) \\ +u_m(\partial_m f^2)e^{4u}+L_{ij}^t u_{li}u_{lj}.\end{aligned}$$

Using (4.6) and Lemma 2 in [31], we obtain

$$\begin{aligned}0\geq T_{ij}\Big(\frac{1-t}{2}\sum_k R_{klkm}u_l u_m\delta_{ij}+R_{iljm}u_l u_m-u_m\partial_m(A_g^t)_{ij}\Big) \\ +u_m(\partial_m f^2)e^{4u}+T_{ij}u_{li}u_{lj}+\frac{1-t}{2}\sum_l T_{ll}u_{ij}u_{ij},\end{aligned}\tag{4.13}$$

where R_{iljm} are the components of the Riemann curvature tensor of g.

Lemma 4.1. *There exists a constant $\beta>0$ such that for $t\in[\delta,1]$,*

$$T_{ij}u_{li}u_{lj}+\frac{1-t}{2}\sum_l T_{ll}u_{ij}u_{ij}\geq\beta\sum_l T_{ll}|\nabla u|^4.\tag{4.14}$$

Remark. This was proved in [27], using the result in [20].

Using the lemma, we have

$$0 \geq T_{ij}\left(\frac{1-t}{2}\sum_k R_{klkm}u_l u_m \delta_{ij} + R_{iljm}u_l u_m - u_m\partial_m (A_g^t)_{ij}\right) + u_m(\partial_m f^2)e^{4u} + \beta\sum_l T_{ll}|\nabla u|^4. \tag{4.15}$$

Since we are assuming u is bounded above, the $|\nabla u|^4$ term dominates, and the proof proceeds as in [20] or [27]. q.e.d.

Remark. For convenience, we would like to present here a simplified proof of Lemma 4.1 which works for $t < 1$. Although the argument breaks down as $t \to 1$, it covers the case $t_0 = 0$, and therefore suffices for proving Theorem 1.4.

To begin, we claim that if $\beta_t' > 0$ is sufficiently small, then for at least one i_0, $u_{i_0 i_0} \geq \beta_t'|\nabla u|^2$. If not, then $u_{ii} < \beta_t'|\nabla u|^2$ for $i = 1 \dots 4$. Since $\Gamma_2 \subset \{\sigma_1 > 0\}$,

$$\Delta u + 2(1-t)(\Delta u) - 2(2-t)|\nabla u|^2 + |\nabla u|^2 + \sigma_1(A_g^t) > 0.$$

Without loss of generality, we may assume that $\sigma_1(A_g^t) \leq \epsilon|\nabla u|^2$, we then have

$$\Big(1 + 2(1-t)\Big)\Delta u + \Big(1 - 2(2-t)\Big)|\nabla u|^2 + \epsilon|\nabla u|^2 > 0.$$

From the assumption, $\Delta u = \sum u_{ii} < 4\beta_t'|\nabla u|^2$, so

$$4\beta_t'\Big(1 + 2(1-t)\Big)|\nabla u|^2 + \Big(1 - 2(2-t)\Big)|\nabla u|^2 + \epsilon|\nabla u|^2 > 0,$$

which is a contradiction for ϵ and β_t' sufficiently small. We then have

$$\begin{aligned} T_{ij}u_{li}u_{lj} + \frac{1-t}{2}\sum_l T_{ll}u_{ij}u_{ij} &\geq \frac{1-t}{2}\sum_l T_{ll}u_{i_0 i_0}^2 \\ &\geq \frac{1-t}{2}(\beta_t')^2\sum_l T_{ll}|\nabla u|^4, \end{aligned} \tag{4.16}$$

so choose $\beta_t = \frac{1-t}{2}(\beta_t')^2$. This completes the proof.

5. Proof of Theorem 1.1

Proposition 5.1. *Let u_t be a C^4 solution of (3.1) for some $\delta \leq t \leq 1$ satisfying $\underline{\delta} < u_t < \overline{\delta}$, and $\|\nabla u_t\|_{L^\infty} < C_1$. Then for $0 < \alpha < 1$, $\|u_t\|_{C^{2,\alpha}} \leq C_2$, where C_2 depends only upon $\underline{\delta}, \overline{\delta}, C_1$, and g.*

Proof. The C^2 estimate follows from the global estimates in [23], or the local estimates [20] and [27]. We remark that the main fact used in deriving these estimates is that $\sigma_2^{1/2}(A^t)$ is a *concave* function of the second derivative variables, which follows easily from the inequality (2.6). Since $f(x) > 0$, the C^2 estimate implies uniform ellipticity, and the $C^{2,\alpha}$ estimate then follows from the work of [25] and [14] on concave, uniformly elliptic equations. q.e.d.

To finish the proof of Theorem 1.1, we use the continuity method. Recall that we are considering the 1-parameter family of equations, for $t \in [\delta, t_0]$,

$$\sigma_2^{1/2}(g^{-1}A^t_{u_t}) = f(x)e^{2u_t}, \tag{5.1}$$

with $f(x) = \sigma_2^{1/2}(g^{-1}A^\delta_g) > 0$, and δ was chosen so that A^δ_g is positive definite. We define

$$\mathcal{S} = \{t \in [\delta, t_0] \mid \exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (5.1) with } A^t_{u_t} \in \Gamma_2^+\}.$$

The function $f(x)$ was chosen so that $u \equiv 0$ is a solution at $t = \delta$. Since A^δ_g is positive definite, and the positive cone is clearly contained in Γ_2^+, $\mathcal{S}$ is nonempty. Let $t \in \mathcal{S}$, and u_t be any solution. From Proposition 2.2, the linearized operator at u_t, $\mathcal{L}^t : C^{2,\alpha}(M) \to C^\alpha(M)$, is invertible. The implicit function theorem (see [17]) implies that $\mathcal{S}$ is open. Note that since $f \in C^\infty(M)$, it follows from classical elliptic regularity theory that $u_t \in C^\infty(M)$. Proposition 3.1 implies a uniform upper bound on solutions u_t (independent of t). We may then apply Proposition 4.1 to obtain a uniform gradient bound, and Lemma 3.2 then implies a uniform lower bound on u_t. Proposition 5.1 then implies that $\mathcal{S}$ is closed, therefore $\mathcal{S} = [\delta, t_0]$. The metric $\tilde{g} = e^{-2u_{t_0}}g$ then satisfies $\sigma_2(A^{t_0}_{\tilde{g}}) > 0$ and $R_{\tilde{g}} > 0$.

We next verify the inequalities (1.7). We decompose A^t into its trace-free and pure-trace components,

$$\begin{aligned} A^t &= A^t - \frac{1}{n}\sigma_1(A^t)g + \frac{1}{n}\sigma_1(A^t)g \\ &\equiv \overset{\circ}{A^t} + \frac{1}{n}\sigma_1(A^t)g. \end{aligned} \tag{5.2}$$

We now associate to A^t the symmetric transformation $\widehat{A^t}$, defined by

$$\widehat{A^t} \equiv -\overset{\circ}{A}{}^t + \frac{1}{n}\sigma_1(A^t)g. \tag{5.3}$$

That is, $\widehat{A^t}$ is the (unique) symmetric transformation which has the same pure-trace component as A, but the opposite trace-free component.

Lemma 5.1. *The tensors $\widehat{A^t}$ and A^t satisfy the equalities*

$$\sigma_1(\widehat{A^t}) = \sigma_1(A^t), \tag{5.4}$$

$$\sigma_2(\widehat{A^t}) = \sigma_2(A^t). \tag{5.5}$$

Proof. The proof of (5.4) is immediate from the definition of $\widehat{A^t}$. To prove (5.5), we use the identity

$$\begin{aligned}\sigma_2(A^t) &= -\frac{1}{2}|A^t|^2 + \frac{1}{2}\sigma_1(A^t)^2 \\ &= -\frac{1}{2}|\overset{\circ}{A}{}^t + \frac{1}{n}\sigma_1(A^t)g|^2 + \frac{1}{2}\sigma_1(A^t)^2.\end{aligned}$$

Since the decomposition (5.2) is orthogonal with respect to the norm $|\cdot|^2$, we conclude

$$\sigma_2(A^t) = -\frac{1}{2}|-\widehat{A^t}|^2 + \frac{1}{2}\sigma_1(A^t)^2 = \sigma_2(\widehat{A^t}).$$

q.e.d.

Combining Lemma 5.1 and Proposition 2.1 we have:

Proposition 5.2. *If the eigenvalues of A^t are in Γ_2^+, then*

$$-A^t + \sigma_1(A^t)g > 0, \text{ and} \tag{5.6}$$

$$A^t + \frac{n-2}{n}\sigma_1(A^t)g > 0. \tag{5.7}$$

Proof. The tensor in (5.6) is simply the first Newton transformation of A^t, which is positive definite by Proposition 2.1. By Lemma 5.1, $\sigma_2(\widehat{A^t}) > 0$ and $\sigma_1(\widehat{A^t}) > 0$. Thus, the eigenvalues of $\widehat{A^t}$ are also in Γ_2^+. By Proposition 2.1, the first Newton tranform $T_1(\widehat{A^t})$ is positive definite. By definition,

$$\begin{aligned}T_1(\widehat{A^t}) &= -\widehat{A^t} + \sigma_1(\widehat{A^t})g \\ &= -\left(-\overset{\circ}{A}{}^t + \frac{1}{n}\sigma_1(A^t)g\right) + \sigma_1(A^t)g \\ &= A^t + \frac{(n-2)}{n}\sigma_1(A^t)g.\end{aligned}$$

q.e.d.

When expressed in terms of Ric and $n = 4$, (5.6) and (5.7) are exactly (1.7).

6. Proof of Theorem 1.4

The assumption (1.14) corresponds to $t_0 = 0$ in Theorem 1.1. From (1.7) we find a conformal metric (which for simplicity, we again denote by g) with $\mathrm{Ric}_g < R_g g$. Theorem 1.4 then follows from:

Proposition 6.1. *If* $\mathrm{Ric}_g \leq R_g g$*, then* $P \geq 0$*, and* $\mathrm{Ker}\, P = \{\text{constants}\}$.

Proof. We again recall that the Paneitz operator is defined by

$$P\phi = \Delta^2\phi + \delta\Big(\frac{2}{3}R_g g - 2\mathrm{Ric}_g\Big)d\phi. \tag{6.1}$$

Integrating by parts,

$$\langle P\phi, \phi\rangle_{L^2} = \int_M \bigg((\Delta\phi)^2 + \frac{2}{3}R_g|\nabla\phi|^2 - 2\mathrm{Ric}_g(\nabla\phi, \nabla\phi)\bigg)\, d\mathrm{vol}_g. \tag{6.2}$$

From the Bochner formula,

$$0 = \int \big(|\nabla^2\phi|^2 + \mathrm{Ric}_g(\nabla\phi, \nabla\phi) - (\Delta\phi)^2\big)\, d\mathrm{vol}_g. \tag{6.3}$$

Substituting (6.3) into (6.2), we have

$$\begin{aligned}
&\langle P\phi, \phi\rangle_{L^2}\\
&= \int_M \bigg(-\frac{1}{3}(\Delta\phi)^2 + \frac{4}{3}(\Delta\phi)^2 + \frac{2}{3}R_g|\nabla\phi|^2 - 2\mathrm{Ric}_g(\nabla\phi, \nabla\phi)\bigg)\, d\mathrm{vol}_g\\
&= \int_M \bigg(-\frac{1}{3}(\Delta\phi)^2 + \frac{4}{3}|\nabla^2\phi|^2 + \frac{2}{3}R_g|\nabla\phi|^2 - \frac{2}{3}\mathrm{Ric}_g(\nabla\phi, \nabla\phi)\bigg)\, d\mathrm{vol}_g\\
&= \int_M \bigg(\frac{4}{3}|\mathring{\nabla}^2\phi|^2 + \frac{2}{3}(R_g g - \mathrm{Ric}_g)(\nabla\phi, \nabla\phi)\bigg)\, d\mathrm{vol}_g\\
&\geq \int_M \frac{4}{3}|\mathring{\nabla}^2\phi|^2 d\mathrm{vol}_g,
\end{aligned}$$

where $\mathring{\nabla}^2\phi = \nabla^2\phi - (1/4)(\Delta\phi)g$. Consequently, $P \geq 0$. Assume by contradiction that $P\phi = 0$, and ϕ is not constant. From the above, we conclude that $\overset{\circ}{\nabla}^2\phi \equiv 0$. By [28, Theorem A], g is homothetic to S^4. We then have

$$0 = \langle P\phi, \phi\rangle_{L^2} = \int_M \frac{4}{3}|\overset{\circ}{\nabla}^2\phi|^2 d\mathrm{vol}_g + \frac{1}{2}R_g \int_M |\nabla\phi|^2 d\mathrm{vol}_g,$$

and therefore $\phi = \text{constant}$. q.e.d.

Remark. In [11], the nonnegativity of the Paneitz operator was shown assuming $Rg - \lambda\mathrm{Ric} \geq 0$ for $\lambda \in (1, 3]$. The above proposition extends this to the endpoint $\lambda = 1$.

Corollary 6.1. *If* $\mathrm{Ric}_g \geq 0$, *then* $P \geq 0$, *and* $\mathrm{Ker}\, P = \{\text{constants}\}$.

Proof. Clearly, $\mathrm{Ric}_g \geq 0$ implies that $\mathrm{Ric}_g \leq R_g g$, so this follows directly from Proposition 6.1. q.e.d.

Remark. The construction in [29] yields metrics with positive Ricci curvature on the connect sums $k(S^2 \times S^2)$, $k\mathbb{CP}^2\#\overline{\mathbb{CP}}^2)$, and $(k+l)\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$. Consequently, from Corollary 6.1, and the results in [10], these manifolds admit metrics with $Q = \text{constant}$.

7. Examples

The following theorem will allow us to give many examples of metrics satisfying the conditions of Theorem 1.4.

Theorem 7.1. *Let* (M, g) *satisfy* $\int Q_g d\mathrm{vol}_g \geq 0$. *If* $Y[g] > 4\sqrt{3k}\pi$, $k < 8$, *then the manifold* $N = M\#k(S^1 \times S^3)$ *admits a metric* $\widetilde{g}$ *satisfying* (1.14). *If* $Y[g] > 8\sqrt{3}\pi$, *then the manifold* $N = M\#l(\mathbb{RP}^4)$ *admits a metric* $\widetilde{g}$ *satisfying* (1.14) *for* $l < 9$. *Consequently, these manifolds* N *admit metrics with* $Q = \text{constant}$.

Proof. From the assumption on $\int Q d\mathrm{vol}$ and the Chern-Gauss-Bonnet formula, we have

$$\int_M |W_g|^2 d\mathrm{vol}_g \leq 8\pi^2\chi(M). \tag{7.1}$$

From [2, Proposition 4.1], given a point $p \in M$, and $\epsilon > 0$, M admits a metric g' so that g' is locally conformally flat in a neighborhood of p, and

$$\begin{aligned} &|Y[g'] - Y[g]| < \epsilon, \\ &\Big|\int_M |W_g|^2 d\mathrm{vol}_g - \int_M |W_{g'}|^2 d\mathrm{vol}_{g'}\Big| < \epsilon. \end{aligned} \tag{7.2}$$

We next put a metric on the connect sum using the technique in [24]. Since g' is locally conformally flat near p, there is a conformal factor on $M - \{p\}$ which makes the metric look cylindrical near p.

For the first case, since $S^1 \times S^3$ is locally conformally flat, for any $p' \in S^1 \times S^3$ there is a conformal factor on $S^1 \times S^3 - \{p'\}$ which makes the metric look cylindrical near p'. Therefore one can put a metric on N by identifying the cylindrical regions together along their boundaries. From the construction in [24], there are sequences of locally conformally flat metrics on $k(S^1 \times S^3)$ whose Yamabe invariants approach $\sigma(k(S^1 \times S^3)) = \sigma(S^4) = 8\sqrt{6}\pi > Y[g']$, where σ denotes the diffeomorphism Yamabe invariant, so we choose a locally conformally flat metric g_1 on $k(S^1 \times S^3)$ satisfying $Y[g_1] \geq 8\sqrt{6}\pi - \epsilon$. We have $min\{Y[g_1], Y[g']\} = Y[g']$, so following the proof [24, Theorem 2], by changing the length of the cylindrical region, one can put a metric $\widetilde{g}$ on the connect sum $N = M \# k(S^1 \times S^3)$ with $Y[\widetilde{g}] > Y[g'] - \epsilon$. Clearly we also have

$$\Big| \int_M |W_g|^2 d\mathrm{vol}_g - \int_N |W_{\widetilde{g}}|^2 d\mathrm{vol}_{\widetilde{g}} \Big| < \epsilon,$$

which along with (7.1) implies

$$\int_N |W_{\widetilde{g}}|^2 d\mathrm{vol}_{\widetilde{g}} \leq 8\pi^2 \chi(M) + \epsilon. \tag{7.3}$$

We next verify that, for appropriate ϵ, the metric $\widetilde{g}$ satisfies the condition (1.14). To see this, write $(Y[g])^2 = 48k\pi^2 + 3\delta$, with $\delta > 0$, and noting that $\chi(N) = \chi(M) - 2k$ we have

$$\begin{aligned} 2\int_N Q_{\widetilde{g}} d\mathrm{vol}_{\widetilde{g}} + \frac{1}{3}(Y[\widetilde{g}])^2 &= 8\pi^2\chi(N) - \int_N |W_{\widetilde{g}}|^2 d\mathrm{vol}_{\widetilde{g}} + \frac{1}{3}(Y[\widetilde{g}])^2 \\ &\geq 8\pi^2\chi(N) - 8\pi^2\chi(M) + \frac{1}{3}(Y[g])^2 - C\epsilon \\ &= -16k\pi^2 + \frac{1}{3}(Y[g])^2 - C\epsilon \\ &= \delta - C\epsilon > 0, \end{aligned}$$

for ϵ sufficiently small.

For the second case, since $\mathbb{RP}^4$ is locally conformally flat, we do exactly the same gluing as before. Again we use [2, Proposition 4.1] to find a metric g' on M satisfying (7.2). We fix the standard metric g_0 on $\mathbb{RP}^4$, which is locally conformally flat. Since $Y([g']) > 8\sqrt{3}\pi = Y[g_0]$ we have $min\{Y[g_0], Y[g']\} = Y[g_0]$, so from the construction in [24], we can put a metric $\widetilde{g}$ on the connect sum $N = M\#l(\mathbb{RP}^4)$ with $Y[\widetilde{g}] > 8\sqrt{3}\pi - \epsilon$, and which also satisfies (7.3).

Write $(Y[g])^2 = 3(64\pi^2 + \delta)$, with $\delta > 0$, and noting that $\chi(N) = \chi(M) - l$,

$$\begin{aligned} 2\int_N Q_{\widetilde{g}} d\mathrm{vol}_{\widetilde{g}} + \frac{1}{3}(Y[\widetilde{g}])^2 &= 8\pi^2\chi(N) - \int_N |W_{\widetilde{g}}|^2 d\mathrm{vol}_{\widetilde{g}} + \frac{1}{3}(Y[\widetilde{g}])^2 \\ &\geq 8\pi^2\chi(N) - 8\pi^2\chi(M) + \frac{1}{3}(Y[g])^2 - C\epsilon \\ &= -8l\pi^2 + 64\pi^2 + \delta - C\epsilon > 0, \end{aligned}$$

for ϵ sufficiently small and $l < 9$. q.e.d.

We next write down some specific examples of (M, g) satisfying the assumptions of Theorem 7.1. We will use the fact that if (M, g) is a positive Einstein manifold, then $\int Q_g d\mathrm{vol}_g > 0$ and the Yamabe invariant is attained by g.

(1) $M = S^2 \times S^2$ with the product metric, $Y[g] = 16\pi > 4\sqrt{3k}\pi$ for $k < 6$, so we have

$$N = (S^2 \times S^2)\#k(S^1 \times S^3),\ k \leq 5. \tag{7.4}$$

(2) $M = \mathbb{CP}^2$ with the Fubini-Study metric, $Y[g] = 12\sqrt{2}\pi$ (see [26]). Since $12\sqrt{2}\pi > 4\sqrt{3k}\pi$ for $k < 6$, this yields the examples

$$N = \mathbb{CP}^2\#k(S^1 \times S^3),\ k \leq 5. \tag{7.5}$$

(3) Again, we take $M = \mathbb{CP}^2$. We have $12\sqrt{2}\pi > 8\sqrt{3}\pi$, so from the second statement in Theorem 7.1 we have

$$N = \mathbb{CP}^2\#k(\mathbb{RP}^4),\ k \leq 8. \tag{7.6}$$

(4) $M = \mathbb{CP}^2\#l\overline{\mathbb{CP}}^2$, $3 \leq l \leq 8$, M admits Kähler-Einstein metrics satisfying $Y[g] = 4\pi\sqrt{2(9-l)}$ (see [22], [26]). Since $4\pi\sqrt{2(9-l)} > 4\sqrt{3}\pi$ for $l < 8$, we have the examples

$$N = \mathbb{CP}^2\#l\overline{\mathbb{CP}}^2\#(S^1 \times S^3),\ 3 \leq l \leq 7. \tag{7.7}$$

(5) $N = k(S^1 \times S^3)\#l(\mathbb{RP}^4)$, $2k + l \leq 9$. We do not need Theorem 7.1 for this example, we argue directly. By the construction in [24], these manifolds admit locally conformally flat metrics $\widetilde{g}$

with $Y[\widetilde{g}] \approx Y[\mathbb{RP}^4, g_0] = 8\sqrt{3}\pi$, We have $\chi(N) = -2k - l + 2$, so the assumption (1.14) is that

$$0 < 8\pi^2(-2k - l + 2) + \frac{1}{3}(Y[g])^2 \approx 8\pi^2(-2k - l + 2) + 64\pi^2,$$

which is satisfied for $2k + l < 10$.

The above examples are all summing with locally conformally flat manifolds, but this is not necessary in our construction. We end with a corollary, whose proof is similar to the proof of Theorem 7.1.

Corollary 7.1. *Let (M_1, g_1) and (M_2, g_2) satisfy $\int_{M_i} Q_{g_i} d\mathrm{vol}_{g_i} \geq 0$, and $Y[g_i] > 4\sqrt{3}\pi$. Then the manifold $N = M_1 \# M_2$ admits a metric $\widetilde{g}$ satisfying (1.14). Consequently, N admits a metric with $Q =$ constant.*

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